ANISOTROPY OF THE MEAN FREE PATH LENGTH IN STATIONARY GAS FLOWS

A. L. Stasenko

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Using the concept of the anisotropy of the mean free path length we study the transition from continuous to rarefied flow for spherical and cylindrical gas sources for rigid and Maxwellian molecules.

We know that when a jet flows into a vacuum (for example, [1-6]), the continuous medium gradually becomes collisionless, while the temperature and the Mach number tend asymptotically to constant "frozen" values. This conclusion is derived from both isotropic (spherically symmetric in the velocity space) [2, 5] and ellipsoidal [3, 4] distribution functions, in the latter case the anisotropy of the flow being described by two temperatures: longitudinal $T_{\|}$ and transverse $T_{\perp}$.

In this paper the anisotropy of the flow is characterized by the fact that the molecules on average traverse different parts of the mean free path in different directions. The concept of the anisotropy of the mean free path length was used previously by Kogan [1] to investigate flow past a flat plate.

The most approximate assumption in this paper is that the distribution function (for the case of rigid molecules) is locally Maxwellian and that the gas-dynamic parameters are the solution of the corresponding Euler equation. But the resulting values of the radius of transition from a continuous medium to a rarefied medium of "frozen" temperature and Mach number are close to the values found by other methods [3, 5, 6] for spherical and cylindrical sources. In addition, the discussion given below can be used for the anisotropy characteristics and to determine the "boundaries" of the continuous flow for a jet of arbitrary shape if we have found all the Maxwellian parameters for the jet.

We define the anisotropy of the mean free path length l of a test molecule using the equation

$$\int_0^l n_1 \langle \sigma g \rangle_1 \frac{dy}{c} = 1, \tag{1}$$

where $dy/c = dx/\xi$ is the differential of the flight time of the molecule in coordinate systems connected respectively with the moving gas or with the source (Fig. 1) and σg is averaged over the velocity space of the field molecules and depends on the velocity c of the test molecule. The latter can be chosen, for example, equal to the mean velocity $\langle c \rangle$ at the point $\bar{r}$ of escape (the point of the "last" collision) or as the velocity of any group of molecules. The anisotropy of the mean free path length introduced by Eq. (1) is a function of the point $\bar{r}$ in the flow, the direction through that point, the flow parameters ($\varkappa$, Kn_0 in the retarded gas, etc.), and the law of intermolecular interaction, since $\sigma \sim g^{-4/(s-1)}$ [7]. In particular, for Maxwellian molecules (s = 5), σg is independent of the relative velocity for any distribution function and it can be taken outside the integral sign in (1), for rigid molecules ($s = \infty$), σ = const and $\langle \sigma g \rangle = \sigma \langle g \rangle$. When n = const, σ = const, and $\langle g \rangle = \sqrt{2}\,\langle c \rangle$ (a homogeneous gas at rest), there follows from (1) the usual expression for an isotropic mean free path $l_i = (\sqrt{2}\,\sigma n)^{-1}$.

By (1), the anisotropy of l becomes marked when the flow parameters in the integrand vary significantly along the mean free path (a similar situation also occurs in a shock wave). In this sense the definition (1) of an anisotropic mean free path length is rather formal, but the formalism is justified by the

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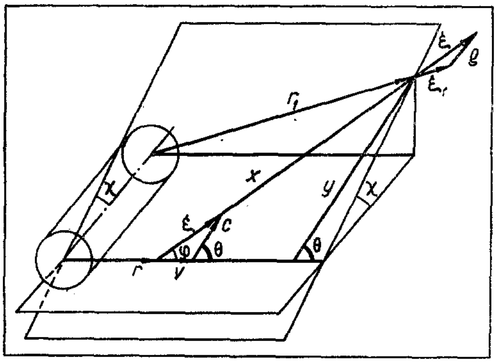

Fig. 1. Computational scheme for cylindrical ($0 \le \chi \le \pi/2$) and spherical ($\chi = \pi/2$) sources.

clarity with which the pattern is visualized and the qualitatively true results, the validity of which is confirmed by comparison with other theories.

Equation (1) defines l for any flow and any distribution function. In what follows, for rigid molecules we take a locally Maxwellian velocity distribution function $f(\bar{r}, \bar{\xi}) = (h/\pi)^{3/2} \exp[-h(\bar{\xi} - \bar{V})^2]$ and by way of examples, consider spherical and cylindrical flows of inviscid nonheat-conducting gas (rigid and Maxwellian molecules), the macroscopic parameters of which, on the "sonic" surface of the sources $R = R_*$ have critical values $V = a_*$, $T = T_*$, $\rho = \rho_*$. At a sufficient distance from these sources ($r \gg 1$) the velocity of the macroscopic motion is constant to order $O(M^{-2})$ (hypersonic approximation):

$$\frac{V}{a_*} = \left(\frac{\varkappa+1}{\varkappa-1}\right)^{1/2}, \quad \left(\frac{T}{T_0}\right)^{\frac{1}{\varkappa-1}} = \frac{\rho}{\rho_0} = \frac{n}{n_0} = \frac{C}{r^\nu}, \tag{2}$$
$$C = \left(\frac{2}{\varkappa+1}\right)^{\frac{1}{\varkappa-1}} \left(\frac{\varkappa-1}{\varkappa+1}\right)^{1/2}, \quad \langle \varepsilon \rangle = \frac{\langle c \rangle}{V} \sim M^{-1}.$$

Spherical Source ($\nu = 2$). Maxwellian Molecules ($s = 5$). In this case for any distribution function, using (2) and the geometrical relations following from Fig. 1 (for $\chi = \pi/2$), the integral (1) can be evaluated in terms of elementary functions and the mean free path can be expressed as

$$\frac{l}{r} = \frac{\sin\theta}{a^2} \operatorname{tg}\left(Q\sin\theta + \operatorname{arctg}\frac{b}{\sin\theta}\right) - \frac{b}{a^2}, \tag{3}$$
$$Q = r\frac{c}{\langle c_0 \rangle}\frac{l_0}{C}, \quad l_0 = \frac{2\mu_0}{\rho_0 \langle c_0 \rangle}.$$

It is easy to see that there is a set of points $r_\infty(\theta)$ such that the test molecule does not experience any further collisions when it leaves them in the direction θ (as a result of its "penultimate" collision). Putting $l = \infty$ in (3), we obtain the following transcendental equation for $r_\infty(\theta)$:

$$\varepsilon_\infty^{-1} = \sin\theta \operatorname{tg}^{-1}(Q_\infty \sin\theta) - \cos\theta. \tag{4}$$

In particular, when $\theta = 0$ and $\theta = \pi$ we have

$$Q_\infty(\varepsilon_\infty^{-1} \pm 1) = 1. \tag{5}$$

Since we are considering a region of flow where $r \gg 1$, we have $\varepsilon_\infty \ll 1$, and the two radii $r_\infty(0)$ and $r_\infty(\pi)$ defined by Eq. (5) differ by an amount $O(\varepsilon_\infty)$; the radius r_Γ lies between them, obtained from the same equation if in it we neglect unity by comparison with ε_∞^{-1}. It is interesting to note that the latter is also obtained from the l-c criterion of [5], based on an isotropic mean free path length $l_i = 2\mu/\rho\langle c \rangle$. In what follows it is useful to write r_Γ for both flows under consideration and an arbitrary power law relation between the viscosity and the temperature $\mu \sim T^\omega$ ($\omega = 1$ for Maxwellian and $\omega = 1/2$ for rigid molecules):

$$r_\Gamma^\zeta = \left(\frac{\pi\varkappa}{8}\right)^{1/2} \left(\frac{\varkappa+1}{2}\right)^\beta \left(\frac{\varkappa+1}{\varkappa-1}\right)^\gamma l_0, \tag{6}$$
$$\zeta = \nu(\varkappa-1)(\omega-1) - \nu + 1, \quad \beta = [2-(\varkappa-1)(2\omega-1)][2(\varkappa-1)]^{-1},$$
$$\gamma = 1/2\,[2-(\varkappa-1)(\omega-1)].$$

Thus, for $r_\Gamma > r \gg 1$, to order $O(\varepsilon)$, we have $Q = \varepsilon \ll 1$, $a = b = \varepsilon^{-1}$, so that (3) can be simplified:

$$l = P\frac{r^{3-\varkappa}}{r_\Gamma - r}, \quad P = \left(\frac{8}{\pi\varkappa}\right)^{1/2} \left(\frac{\varkappa-1}{\varkappa+1}\right)^{\frac{\varkappa+1}{4}}.$$

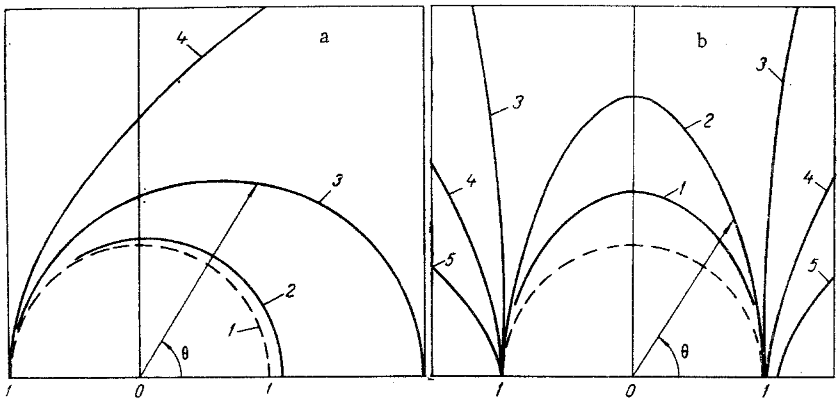

Fig. 2. Meridional sections of the $l(r, \theta)$-surfaces normalized by $l(r, \pi)$: a) spherical source; Maxwellian molecules, Eq. (3); $\varkappa = 5/3$, $l_0 = 10^{-5}$, $r_\Gamma = 23{,}176$; 1) $r/r_\Gamma = 0$–0.9; $l(r, \pi) = 120$; 2) 0.99 and 1231, respectively; 3) 0.999 and 6327; 4) 1.0 and 11,581; b) rigid molecules, Eq. (8), $r_\Gamma = 333$; 1) $r/r_\Gamma = 0.5$; $l(r, \pi) = 1.23$; 2) 0.6 and 2.07, respectively; 3) 0.7 and 3.58; 4) 0.8 and 6.59; 5) 0.9 and 15.2.

From this we see that in the case of Maxwellian molecules far from the source the l-surfaces are nearly spheres of radius increasing more rapidly than for an isotropic mean free path $l_i \sim r^{3-\varkappa}$. When r is close to r_Γ, the l-surfaces are strongly elongated in the direction of the flow and, finally, when $r = r_\infty(0) \approx r_\Gamma$, the l-radius corresponding to $\theta = 0$ becomes infinitely large (Fig. 2a).

Rigid Molecules ($s = \infty$). In this case $\langle \sigma g \rangle_1 = \sigma \langle g \rangle_1$. To simplify, we assume further that the field molecules do not have thermal velocities, $c_1 \equiv 0$ (more exact calculation shows that this assumption introduces a comparatively small error; in general it does not affect the position of the boundary of the continuous medium). Then $\langle g \rangle_1 = g_1$, where

$$\left(\frac{g_1}{c}\right)^2 = 1 + \frac{2}{\varepsilon^2} - \frac{2}{\varepsilon}\left(\cos\theta - \frac{dr_1}{dy}\right) \leqslant 1, \tag{7}$$

and (1) can be expressed in terms of the elliptic integral of the second kind $E(\delta, q)$:

$$\begin{gathered}\frac{l_0}{2C} r\sin\theta = \left[(a+\varepsilon^{-1}) E(\delta, q) - \frac{2a\sin\alpha}{3\sqrt{a^2+\varepsilon^{-2}-2a\varepsilon^{-1}\cos\alpha}}\right]_{\alpha_l}^{\alpha^0},\\ \sin\delta = \frac{a+\varepsilon^{-1}}{\sqrt{2}}\sqrt{\frac{1-\cos\alpha}{a^2+\varepsilon^{-2}-2a\varepsilon^{-1}\cos\alpha}},\quad q = \frac{2}{a+\varepsilon^{-1}}\sqrt{\frac{a}{\varepsilon}},\\ a\cos\alpha = \frac{a^2(y/r)+b}{\sqrt{a^2(y/r)^2+2b(y/r)+1}},\quad l_0 = \frac{1}{n_0\sigma_0}.\end{gathered} \tag{8}$$

The limits of integration α^0 and α_l correspond to the conditions $y = 0$ and $y = l$; in particular $\cos\alpha^0 = b/a$, $\sin\delta^0 = (a+\varepsilon^{-1})\sqrt{(a-b)/2a}$. Putting $l(r, \theta) = \infty$ ($\alpha_l = \delta_l = E(\delta_l, q) = 0$), from (8) we find an equation for the lines $r_\infty(\theta)$:

$$\frac{\sin\theta}{a+\varepsilon^{-1}}\left(\frac{l_0 r_\infty}{2C} + \frac{2}{\varepsilon_\infty}\right) = E(\delta_\infty^0, q). \tag{9}$$

In particular, when $\theta = 0$ and $\theta = \pi$, from this we obtain an equation similar to (5) defining the two radii $r_\infty(0)$ and $r_\infty(\pi)$ which differ from each other and from r_Γ (cf. (6) when $\nu = 2$ and $\omega = 1/2$) by an amount $O(\varepsilon_\infty)$. Near $\theta = \pi/2$, $r_\infty(\theta)$ has a minimum equal, to order $O(\varepsilon^2)$, to $r_m = r_\Gamma/2^{1/\varkappa}$.

Thus, we can take $\Delta = r_\Gamma(1 - 2^{-1/\varkappa})$ as the scale in determining the width of the transition region.

Figure 2b gives the meridional sections of the l-surfaces (6), the radii-vectors of each for convenience being normalized by the corresponding value of $l(r, \pi)$ and drawn through the same center. Indeed, as r increases, they strongly increase in dimension and elongate in the direction perpendicular to the flow, due to the macroscopic dispersion of the molecules $\overline{V}_1 - \overline{V}$ and the associated reduction in the

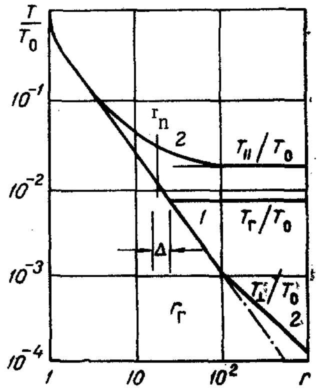

Fig. 3. The temperature as a function of the radius according to the proposed theory (curve 1) and according to the solution of the model Boltzmann equation (curves 2 [3]) (spherical source, rigid molecules, $\varkappa = 5/3$, $l_0 = 1.47 \cdot 10^{-3}$).

relative velocity g. Finally, when $r = r_m$ they "split": one of the radii becomes infinitely large. But in directions near to the direction of the streamline, collisions continue to occur and terminate when $r \approx r_\Gamma$. When $r > r_\Gamma$, the temperature and Mach number become constant and equal to the "frozen" values T_Γ and M_Γ. In Fig. 3, T_Γ is compared with the solution of the Boltzmann equation obtained in [3]. But here we use the "transition" radius r_n [3]. There is a comparison in [5] of M_Γ and the experimental results of [6].

We note that the radius at which the inertia forces become comparable with the viscosity forces is, according to the estimate in [8], greater than r_Γ which justifies the above discussion using the macroscopic Euler equations.

Cylindrical Source ($\nu = 1$). In this case, as distinct from the spherical, the l-surfaces are not bodies of rotation and depend on the angle of incidence χ of the plane containing V and c, $l = l(r, \theta, \chi)$ (Fig. 1).

Maxwellian Molecules (s = 5). Integrating (1), we obtain the following quadratic equation for l:

$$a\left[1+2b\left(\frac{l}{r}\right)+a^2\left(\frac{l}{r}\right)^2\right]^{1/2}+a^2\left(\frac{l}{r}\right)+b=(a+b)\exp\left(\frac{acl_0}{\langle c_0\rangle C}\right). \tag{10}$$

Rigid Molecules ($s = \infty$). In this case we have the following equation for l (when $c_1 \equiv 0$):

$$2a\,\frac{l_0}{C}=\left[\sqrt{A-Ba}\ln\frac{\sqrt{A-Bz}+\sqrt{A-Ba}}{\sqrt{A-Bz}-\sqrt{A-Ba}}+\sqrt{A+Ba}\ln\frac{\sqrt{A-Bz}-\sqrt{A+Ba}}{\sqrt{A-Bz}+\sqrt{A+Ba}}\right]_b^{z_l},$$

$$z=\frac{dr_1}{dy}=\frac{a^2(y/r)+b}{\sqrt{1+2b(y/r)+a^2(y/r)^2}}. \tag{11}$$

Using (10) and (11) we can show that l does not become infinite for any value of its arguments, i.e., as distinct from the spherical case, the collisionless regime is impossible in the case of a cylindrical source (a similar conclusion was obtained in [2]). However there is a typical radius r_Γ, defined also from the l-c criterion [5], such that when $1 \ll r \ll r_\Gamma$, l is proportional to r and when $r \gg r_\Gamma$ it increases more sharply (exponentially) with r. But when $r > r_\Gamma$, the Euler equations become meaningless since the effect of viscosity appears [8] and the flow may become subsonic across the cylindrical shock wave [9, 10]. Using (1) we can construct l-surfaces for $r > r_\Gamma$ if we know the solution of the macroscopic hydrodynamical equations.

Thus, the theory of the anisotropic mean free path length yields richer information on the flow structure than the l-c criterion [5] (which defines only the "boundary" of the continuous medium) and in this sense it is a useful development of the isotropic mean free path, while remaining still quite simple.

NOTATION

σ, m — are the collision cross section and mass of molecule;

$\bar{c}$; $\bar{\xi}$; $g = |\bar{\xi}_1 - \bar{\xi}|$ — are the thermal, total, and relative velocities of molecule;

n, ρ, T, V, μ — are the numerical and mass densities, temperature, macroscopic velocity and viscosity of the gas;

a — is the speed of sound;

$\varkappa$ is the ratio of specific heats;
k is Boltzmann's constant.
$h = m/2kT$;
L is the mean free path;
R, φ are the coordinate system fixed to source;
Y, θ are the coordinate system fixed in the moving gas;
$r = R/R_*$;
$y = Y/R_*$;
$l = L/R_* = Kn$ is the Knudsen number;
R_* is the radius of source.

Subscripts

0, *, 1 denote respectively the retarded gas, the surface of the source, and the current point on the trajectory of the test molecule;
s denotes the exponent of the power in the equation for the interaction forces as a function of the distance between the molecules;
$\langle\ \rangle$ denotes the averaging over the velocity space;
$M = V/a$;
$\varepsilon = c/V$;
$a^2 = b^2 + \sin^2\theta \cdot \sin^2\chi$;
$b = \cos\theta + 1/\varepsilon$ (for a spherical source $\chi \equiv \pi/2$).

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